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BIANGULAR COORDINATES.

By G. B. M. ZERR, Philadelphia, Pa.

It is the purpose of this discussion to set forth a very elementary exposition of a most interesting subject* rather than anything new, with the hope that it may be further developed in future issues. In what follows the notation of Professor Genese is used.†

THE STRAIGHT LINE.

We assume two fixed points A, B as poles, and determine the position of any point P in the plane of AB by the angles $PAB=\theta$ and $PBA=\phi$, regarding θ, ϕ as positive on one side of AB and negative on the other.

Let PD be a straight line intersecting AB in D , N the foot of the perpendicular from P on AB ; also let $AB=c, AD=d, BD=f, \cot \theta=\lambda, \cot \phi=\mu$.

Then $AN+BN=c\dots (1)$.

$PN=DN\tan D=(AN-d)\tan D=NB\tan \phi\dots (2)$.

$PN=(f-NB)\tan D=AN\tan \theta\dots (3)$.

Eliminating AN, NB from (1), (2), (3) we get

$$f\tan D\tan \phi - c\tan \theta \tan \phi - (c-f)\tan D\tan \theta = 0,$$

$$\text{or } f\tan D\tan \phi - c\tan \theta \tan \phi - d\tan D\tan \theta = 0.$$

Hence $f\cot \theta - d\cot \phi - c\cot D = 0$. Writing p for f, q for $-d, r$ for $-c\cot D$, we get for the equation to any straight line

$$\text{I. } p^\lambda + q^\mu + r = 0.$$

Hence $\lambda=0, \mu=0$ represents straight lines perpendicular to AB , through A and B , respectively. Let $p_1^\lambda + q_1^\mu + r_1 = 0$ be any other line, α the angle line I makes with this last line. Then, since $r = -c\cot D, r_1 = -c\cot D_1$,

$$\cot \alpha = \frac{(r/c)(r_1/c) + 1}{r/c - r_1/c} = \frac{rr_1 + c^2}{c(r - r_1)}.$$

Therefore $\cot \alpha = 0$ or ∞ according as the lines are perpendicular or parallel. If $r_1 = -c^2/r$, the lines are perpendicular. If $r_1 = r$, the lines are parallel.

*A consideration of biangular coordinates may be found in various sources, including the following: *Quarterly Journal of Mathematics*, Volumes IX and XIII; Carr's *Synopsis of Pure Mathematics*; Milne's *Companion to Weekly Problem Papers*.

†See *Weekly Problem Papers*.

Let AB be the axis of abscissas, and the perpendicular bisector of AB , the axis of ordinates, then, it follows at once that $\lambda = (c+2x)/2y$, $\mu = (c-2x)/2y$.

$$\text{Hence, } x = \frac{c}{2}(\lambda - \mu) / (\lambda + \mu), \quad y = c / (\lambda + \mu) \dots (4).$$

The values of x and y from (4) in the general Cartesian equation gives

$$(Ac + 2C)\lambda + (2C - Ac)\mu + 2Bc = 0.$$

This is of the form

$$p\lambda + q\mu + r = 0, \text{ the same as I.}$$

(4) is used to transform from rectangular to biangular coordinates.

$$\left. \begin{array}{l} p\lambda + q\mu + r = 0 \\ p\lambda_1 + q\mu_1 + r = 0 \\ p\lambda_2 + q\mu_2 + r = 0 \end{array} \right\} \text{ represent the same line.}$$

Eliminating p, q, r , we get

$$\text{II. } \lambda - \lambda_2 = \frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2} (\mu - \mu_2)$$

for the equation to the line joining $(\lambda_1, \mu_1), (\lambda_2, \mu_2)$.

As a simple illustration let it be required to find the locus of the vertex of a triangle when the base and the difference of the squares of the other sides are given.

Here $AB = c$, $AP^2 - BP^2 = m^2$. But $AP^2 - BP^2 = AN^2 - BN^2 = c(AN - BN) = m^2$. Also $\lambda = AN/PN$, $\mu = BN/PN$.

Hence, $c(\lambda - \mu)/PN = m^2$, $PN = c/(\lambda + \mu)$.

Therefore, $(c^2 - m^2)\lambda = (c^2 + m^2)\mu$, a straight line perpendicular to AB , and dividing it in the ratio $(c^2 + m^2) : (c^2 - m^2)$.

THE CONIC SECTIONS.

(a) By (4) the rectangular equation $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ transforms into the biangular equation

$$b^2(4a^2 - c^2)(\lambda^2 + \mu^2) + 2b^2(4a^2 + c^2)\lambda\mu \mp 4a^2c^2 = 0 \dots (5).$$

When $c = 2a$, (5) becomes $\lambda\mu = \pm a^2/b^2 \dots (6)$, an ellipse or hyperbola, according as we use the plus sign or the minus sign, with AB as major axis.

When $c = 2ae$, (5) becomes

$$b^2(1 - e^2)(\lambda^2 + \mu^2) + 2b^2(1 + e^2)\lambda\mu \mp 4a^2e^2 = 0,$$

$$\text{or } \lambda^2 + \mu^2 + \frac{2(1+e^2)}{1-e^2} \lambda\mu = \pm \frac{4a^2 e^2}{b^2(1-e^2)} \dots (7),$$

the equation for the ellipse and the hyperbola, when A, B are the foci.

If $a=b$, (6) becomes $\lambda\mu=1$ for the circle and $\lambda\mu=-1$ for the rectangular hyperbola, and (7) becomes $\lambda+\mu=0$, the line at infinity. In this last case A and B coincide and hence $c=0$. (6) is found geometrically as follows:

The geometry of the ellipse and the hyperbola gives us $PN^2:AN.NB = b^2:a^2$. As before stated, $\mu=NB/PN$, $\lambda=\pm AN/PN$.

Hence $\lambda\mu=\pm AN.NB/PN^2=\pm a^2/b^2$. From II, the secant through two points on (6) is $\lambda-\lambda_2 = \frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}(\mu-\mu_2)$.

But $\lambda_1\mu_1=\lambda_2\mu_2$ or $\lambda_1/\lambda_2=\mu_2/\mu_1$. Hence, $(\lambda_1-\lambda_2)/\lambda_2=-(\mu-\mu_2)/\mu_1$, or $\lambda-\lambda_2=-(\lambda_2/\mu_1)(\mu-\mu_2)$.

For the tangent at (λ_1, μ_1) we have $\lambda_1=\lambda_2$, $\mu_1=\mu_2$, and hence $\lambda-\lambda_1=-(\lambda_1/\mu_1)(\mu-\mu_1)$, or $\lambda/\lambda_1+\mu/\mu_1=2\dots(8)$.

For the circle $\lambda_1\mu_1=1$ and the tangent is given by $\lambda\mu_1+\mu\lambda_1=2$.

(b) Let c be the distance from the focus B to the directrix A of the ellipse or the hyperbola, then if the mid-point of AB is the origin, the rectangular equation is

$$y^2 + (x - \frac{1}{2}c)^2 = e^2(x + \frac{1}{2}c)^2$$

which reduces to

$$4y^2 + 4x^2(1-e^2) - 4cx(1+e^2) + c^2(1-e^2) = 0\dots(9).$$

The substitution of (4) in (9) gives $e^2\lambda^2=\mu^2+1\dots(10)$.

This result is found geometrically as follows: P is a point on the curve, R a point on the directrix, PR is parallel to BA .

Then $BP=ePR=ePA\cos\theta$. Then $BP/PA=\sin\theta/\sin\phi=e\cos\theta$, or $\text{cosec}\phi=e\cot\theta$. Hence, $1+\mu^2=e^2\lambda^2$.

For two points on this curve we easily get $e^2\lambda_1^2-\mu_1^2=e^2\lambda_2^2-\mu_2^2=1$, or $\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2} = \frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}$. Hence, $\lambda-\lambda_2 = \frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}(\mu-\mu_2)\dots(11)$, is the secant through (λ_1, μ_1) , (λ_2, μ_2) . For the tangent $\lambda_1=\lambda_2$, $\mu_1=\mu_2$.

Hence, $\lambda-\lambda_1 = \frac{\mu_1}{e^2\lambda_1}(\mu-\mu_1)$ is the tangent at (λ_1, μ_1) . This further reduces to $e^2\lambda\lambda_1-\mu\mu_1=e^2\lambda_1^2-\mu_1^2=1$. That is, $e^2\lambda\lambda_1-\mu\mu_1=1$ is the tangent.

(c) If AB is a chord of a circle, P the angle inscribed opposite AB , then $\theta+\phi=\pi-P$, $\cot(\theta+\phi)=-\cot P$.

Hence $\lambda\mu+(\lambda+\mu)\cot P=1$ is the equation to the circum-circle of APB .

If $P=\frac{1}{2}\pi$, $\cot P=0$, and $\lambda\mu=1$, as before stated.

(d) Let $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$ be the rectangular equation for the general conic. Substituting the values of x and y from (4) this reduces to the form

$$\text{III. } a\lambda^2 + 2h\lambda\mu + b\mu^2 + 2g\lambda + 2f\mu + e = 0.$$

The tangent to this conic at the point (λ_1, μ_1) is

$$\text{IV. } a\lambda_1\lambda + h(\lambda_1\mu + \mu_1\lambda) + b\mu_1\mu + g(\lambda + \lambda_1) + f(\mu + \mu_1) + e = 0.$$

If $a=b=0=g=f$, III becomes $\lambda\mu + \text{a constant} = 0$; IV becomes $\lambda_1\mu + \mu_2\lambda + \text{a constant} = 0$.

If $a=b=0$, we get the conic referred to any chord. Its equation is of the form

$$\text{V. } \lambda\mu + g_1\lambda + f_1\mu + e_1 = 0.$$

Its tangent at (λ_1, μ_1) is of the form

$$\text{VI. } \lambda_1\mu + \mu_1\lambda + g_1(\lambda + \lambda_1) + f_1(\mu + \mu_1) + e_1 = 0.$$

V and VI can be written as follows:

$$\begin{aligned} (\lambda + f_1)(\mu + g_1) &= \text{a constant}, \\ (\lambda + f_1)(\mu_1 + g_1) + (\lambda_1 + f_1)(\mu + g_1) &= \text{a constant} \\ &= 2(\lambda_1 + f_1)(\mu_1 + g_1). \end{aligned}$$

Hence, $\frac{\lambda + f_1}{\lambda_1 + f_1} + \frac{\mu + g_1}{\mu_1 + g_1} = 2$ is a neater form for VI.

AN APPLICATION.

(e) As an application of biangular coordinates, suppose we have given the base AB of a triangle and the locus of the vertex, to find the locus of the symmedian point K .

Let the vertex C describe the straight line, $p\cot A + q\cot B + r = 0 \dots (13)$.

Let $\cot A = P$, $\cot B = Q$. Since the median AM and the symmedian AK are equally inclined to the bisector of A ,

$$\angle MAB + \angle KAB = \angle A, \text{ or } \cot KAB = \lambda = \frac{\cot A \cot MAB + 1}{\cot MAB - \cot A},$$

$$\text{and } \lambda = \frac{2P^2 + PQ + 1}{P + Q}, \quad \mu = \frac{2Q^2 + PQ + 1}{P + Q}.$$

$$(P + Q)(\lambda - \mu) = 2(P^2 - Q^2), \text{ or } P = \frac{\lambda - \mu}{2} + Q, \quad Q = P - \frac{\lambda - \mu}{2}.$$

The value of Q substituted in the value of λ gives

$$6P^2 - (5\lambda - \mu)P = \mu\lambda - \lambda^2 - 2, \text{ or } P = \frac{5\lambda - \mu \pm \sqrt{(\lambda^2 + 14\lambda\mu + \mu^2 - 48)}}{12}.$$

$$\text{Similarly, } Q = \frac{5\mu - \lambda \pm \sqrt{(\lambda^2 + 14\lambda\mu + \mu^2 - 48)}}{12}.$$

These values of P and Q substituted in (13) gives for the required locus

$$\begin{aligned} & [(5p - q)\lambda + (5q - p)\mu + 12r]^2 = (p + q)^2 (\lambda^2 + 14\lambda\mu + \mu^2 - 48), \\ \text{or } & (2p^2 - pq)\lambda^2 - 2(p^2 + q^2 - pq)\lambda\mu + (2q^2 - pq)\mu^2 \\ & + 2r(5p - q)\lambda + 2r(5q - p)\mu + 12r^2 + 4(p + q)^2 = 0. \end{aligned}$$

That is, $a\lambda^2 + 2\lambda\mu h + b\mu^2 + 2g\lambda + 2f\mu + e = 0$, which is the equation to the general conic.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

327. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

The coefficients of the algebraical equation $f(x) = 0$ are all integers. Show that if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let the equation be $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$.

Then by hypothesis, a_n , and also $(a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n)$ are odd numbers. By subtraction, $(a_0 + a_1 + \dots + a_{n-1})$ is an even number; hence, either all these coefficients are even numbers, or there is an even number of odd coefficients.

Suppose (I) that a_0, \dots, a_{n-1} are all even. Then the substitution for x of *any* integer will give an even result for these terms, which cannot combine with the odd a_n to vanish.

Suppose (II) that an even number of coefficients a_0, \dots, a_{n-1} are odd integers. The substitution of any even integer for x cannot satisfy the equation, just as in (I). Any power of an odd integer (substituted for x) is odd, so that for each odd coefficient, one odd term is obtained. But there is an even number of these, besides a_n . Thus it is again impossible to have the equation satisfied by an integer.